Bresenham's Algorithm

Lecture 31

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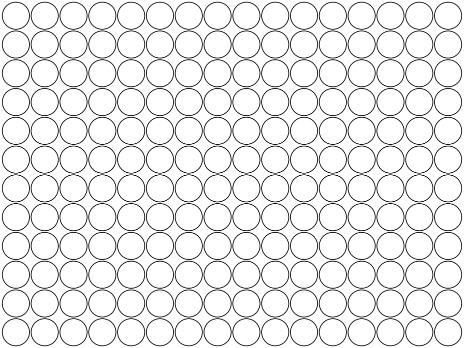
Outline

1. Pixel Coordinates
2. Bresenham's Midpoint Algorithm
   - The Decision Function
   - The Algorithm
   - Examples
3. Assignment

Outline

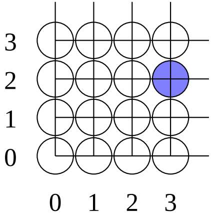
Pixels and Screen Coordinates

- We think of the viewport as consisting of a rectangular array of pixels.

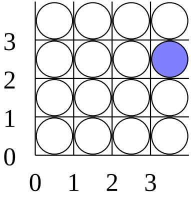


Pixels and Screen Coordinates

- A basic question is, what are the coordinates of the pixels?
- Are they integers, e.g., $(3, 2)$?
- Or, are they half-integers, e.g. $\left(3\frac{1}{2}, 2\frac{1}{2}\right)$?



Integers



Half-integers

Pixels and Screen Coordinates

- OpenGL adopts the view that they are half-integers.
- We will conform to OpenGL's view, so that our results will match OpenGL's results.

Outline

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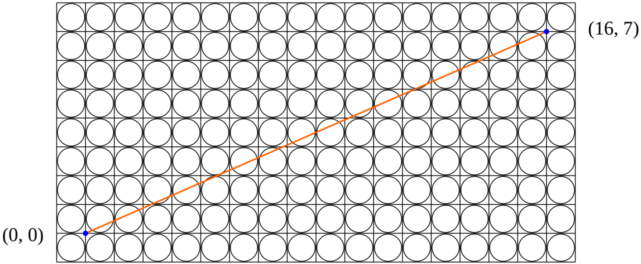
2. Bresenham's Midpoint Algorithm
   - The Decision Function
   - The Algorithm
   - Examples
3. Assignment

Rasterizing Lines

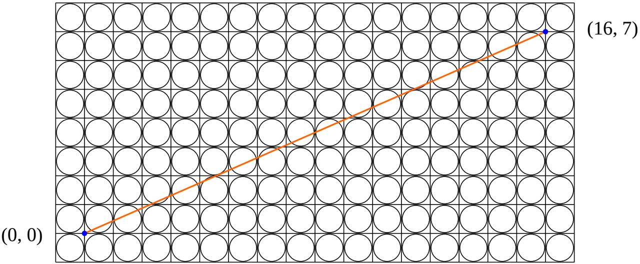
- All lines will go from one intersection of grid lines to another.



Rasterizing Lines

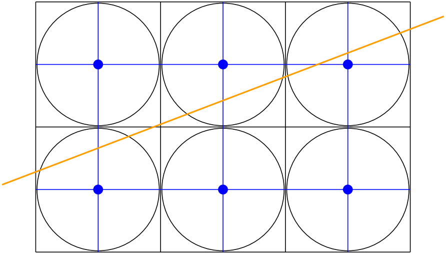
- In what follows, we will assume that the line has a slope between 0 and 1.
- That is, the change in x is greater than the change in y.
- The other cases are “easily” adapted from this case.

Rasterizing Lines



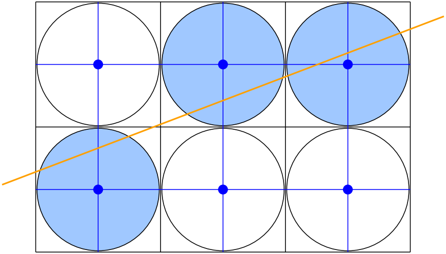
When drawing a line on the screen, how do we decide which pixels to color?

Rasterizing Lines



In each vertical column, choose the pixel whose center is closest to the line.

Rasterizing Lines



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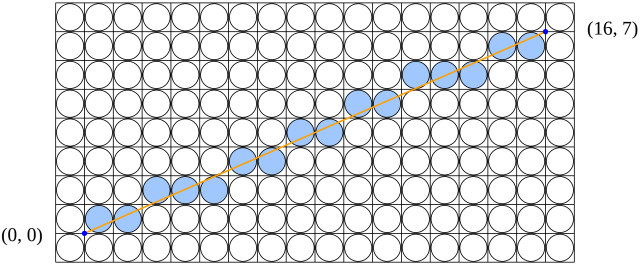
Rasterizing Lines



In each vertical column, choose the pixel whose center is closest to the line.

Choosing the Pixel

- Let the equation of the line be

$$y = mx + b.$$

- Given that we have just colored pixel (p_x, p_y), the next pixel to color is either $(p_x + 1, p_y)$ or $(p_x + 1, p_y + 1)$.
- Why?

Choosing the Pixel

- What is the most efficient way to determine which point to color?
- Based on the equation of a line $y = mx + b$, we could compute $m(p_x + 1) + b$ and then compare it to p_y and $p_y + 1$ to see which is closer.
- Why is this not a good idea?

The Decision Function

- We need a function which, when evaluated, will tell us whether to increment y or to keep y the same.
- Let the line go from $A = (a_x, a_y)$ to $B = (b_x, b_y)$.
- The equation of the line can be written

$$y - a_y = m(x - a_x)$$
$$y - a_y = \left(\frac{b_y - a_y}{b_x - a_x}\right)(x - a_x),$$

or

$$(b_x - a_x)(y - a_y) = (b_y - a_y)(x - a_x).$$

Outline

The Decision Function

- Let $W = b_x - a_x$ (the width).
- Let $H = b_y - a_y$ (the height).
- Note that $W \geq H \geq 0$ by our assumption.
- Then the equation can be written

$$-W(y - a_y) + H(x - a_x) = 0.$$

- Let

$$F(x, y) = -2W(y - a_y) + 2H(x - a_x).$$

- The factor of 2 is introduced to avoid the fraction $\frac{1}{2}$ later.

The Decision Function

- We will call $F(x, y)$ the decision function.

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The Decision Function

- We will call $F(x, y)$ the decision function.
- Note that if we increment x by 1, then $F(x, y)$ *increases* by $2H$.
- And if we increment y by 1, then $F(x, y)$ *decreases* by $2W$.
- And if we do both, then $F(x, y)$ *decreases* by $2(W - H)$.

The Decision Function

- Note that
  - If $F(x, y) < 0$, then the line is below (x, y).
  - If $F(x, y) = 0$, then the line passes through (x, y).
  - If $F(x, y) > 0$, then the line is above (x, y).

Bresenham's Midpoint Algorithm

- Suppose we have shaded pixel $P = (p_x, p_y)$ (centered at $(p_x + \frac{1}{2}, p_y + \frac{1}{2})$).
- Now, in the next column, we need to decide whether to shade the
  - The lower pixel $L = (p_x + 1\frac{1}{2}, p_y + \frac{1}{2})$, or
  - The upper pixel $U = (p_x + 1\frac{1}{2}, p_y + 1\frac{1}{2})$.
- Let M be the point midway between U and L.
- Then $M = \left(p_x + 1\frac{1}{2}, p_y + 1\right)$.

- If the line is above M, then we should color U.
- If the line is below M, then we should color L.
- That is,
  - If $F\left(p_x + 1\frac{1}{2}, p_y + 1\right) > 0$, then color U.
  - If $F\left(p_x + 1\frac{1}{2}, p_y + 1\right) < 0$, then color L.

- In the first column of pixels, the midpoint is $M_1 = \left(a_x + \frac{1}{2}, a_y + 1\right)$.
- The line passes below M_1.
- Indeed, the line passes below the center of the pixel $\left(a_x + \frac{1}{2}, a_y + \frac{1}{2}\right)$.
- For that first midpoint M_1,

$$F_1 = F\left(a_x + \frac{1}{2}, a_y + 1\right) = -2W + H < 0.$$

Bresenham's Midpoint Algorithm

- In the second column of pixels, the midpoint is $M_2 = \left(a_x + 1\frac{1}{2}, a_y + 1\right)$.
- The value of F is

$$\begin{aligned} F_2 &= F\left(a_x + 1\tfrac{1}{2}, a_y + 1\right) \\ &= F_1 + 2H \\ &= 3H - 2W, \end{aligned}$$

  which may be positive or negative.
- We compute it, test it, and decide whether to shade U or L.

Bresenham's Midpoint Algorithm

- To find a recursive formula for the decision, consider what happens as we go from pixel $P_i = (p_x, p_y)$ to pixel P_{i+1}.
- That is, from $x = p_x + \frac{1}{2}$ to $x = p_x + 1\frac{1}{2}$.
- There are two cases:
  - We shaded L in the previous step.
  - We shaded U in the previous step.

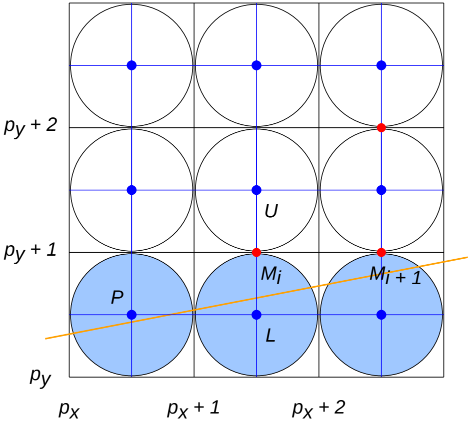
Bresenham's Midpoint Algorithm

- Suppose first that we shaded L.
  - Then the midpoint considered was $M_i = \left(p_x + \frac{1}{2}, p_y + 1\right)$ and $F_i < 0$.
  - Because we shaded L, the next midpoint to consider is $M_{i+1} = \left(p_x + 1\frac{1}{2}, p_y + 1\right)$.
  - We calculate

$$F_{i+1} = F\left(p_x + 1\tfrac{1}{2}, p_y + 1\right) = F_i + 2H,$$

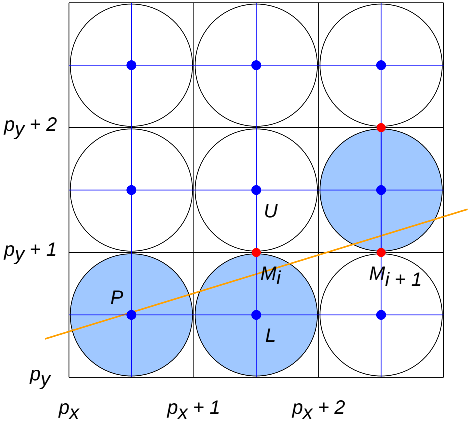
which may be positive or negative.

Bresenham's Midpoint Algorithm



Right - Right

Bresenham's Midpoint Algorithm



Right - Up

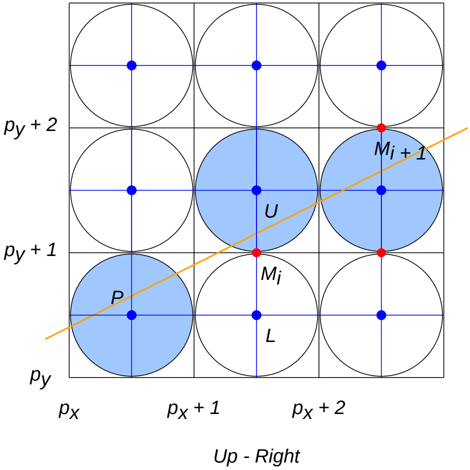
Bresenham's Midpoint Algorithm

- Now suppose that we shaded U.
  - Then the midpoint considered was $M_i = \left(p_x + \frac{1}{2}, p_y + 1\right)$ and $F_i > 0$.
  - Because we shaded U, the next midpoint to consider is $M_{i+1} = \left(p_x + 1\frac{1}{2}, p_y + 2\right)$.
  - We calculate

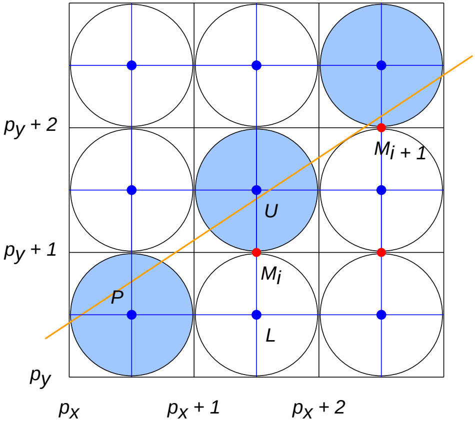
$$F_{i+1} = F\left(p_x + 1\tfrac{1}{2}, p_y + 2\right) = F_i - 2(W - H),$$

    which may or may not be positive.

Bresenham's Midpoint Algorithm



Bresenham's Midpoint Algorithm



Up - Up

Outline

Bresenham's Midpoint Algorithm

- Summarizing the two cases:
  - If the previous value of F was negative, then the next value is $2H$ larger.
  - If the previous value of F was positive, then the next value is $2(W - H)$ smaller.
- This allows us to move quickly from one decision to the next, since we need only add or subtract a constant.
- We do not need to evaluate $F(x, y)$ each time.

Bresenham's Midpoint Algorithm

- To start the process, the first midpoint to be considered is

$$M_2 = \left(a_x + 1\tfrac{1}{2}, a_y + 1\right).$$

- The value of $F\left(a_x + 1\frac{1}{2}, a_y + 1\right)$ is $3H - 2W$.
- Therefore, the initial value of the decision function is $3H - 2W$.

Bresenham's Midpoint Algorithm

Bresenham's Midpoint Algorithm

- Initialize $x = a_x$, $y = a_y$, $F = 3H - 2W$.
- Repeat until $x = b_x$.
  - Increment x.
  - If $F < 0$, then
    - Add $2H$ to F.
  - Else if $F > 0$, then
    - Subtract $2(W - H)$ from F.
    - Increment y.

Bresenham's Midpoint Algorithm

- This algorithm produces the coordinates of the grid point to the lower left of the pixel to be shaded.

Outline

Example

- Rasterize the line from $(0, 0)$ to $(16, 7)$.
  - $W = 16$, $H = 7$.
  - $2H = 14$, $2(W - H) = 18$, $3H - 2W = -11$.

x	F	Δy	y	x	F	Δy	y	x	F	Δy	y
0				6				12			
1				7				13			
2				8				14			
3				9				15			
4				10				16			
5				11							

Example

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x	F	Δy	y	x	F	Δy	y	x	F	Δy	y
0	-11			6	9			12	-3		
1	3			7	-9			13	11		
2	-15			8	5			14	-7		
3	-1			9	-13			15	7		
4	13			10	1			16	-11		
5	-5			11	-17						

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x	F	Δy	y	x	F	Δy	y	x	F	Δy	y
0	-11	0		6	9	1		12	-3	0	
1	3	1		7	-9	0		13	11	1	
2	-15	0		8	5	1		14	-7	0	
3	-1	0		9	-13	0		15	7	1	
4	13	1		10	1	1		16	-11	0	
5	-5	0		11	-17	0					

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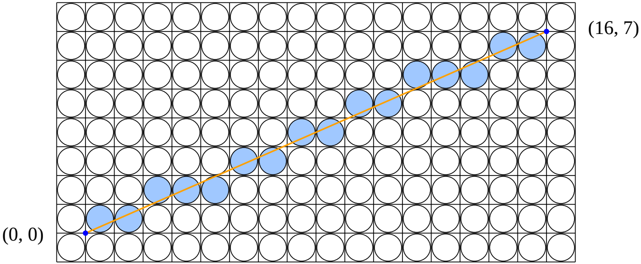
x	F	Δy	y	x	F	Δy	y	x	F	Δy	y
0	-11	0	0	6	9	1	2	12	-3	0	5
1	3	1	0	7	-9	0	3	13	11	1	5
2	-15	0	1	8	5	1	3	14	-7	0	6
3	-1	0	1	9	-13	0	4	15	7	1	6
4	13	1	1	10	1	1	4	16	-11	0	7
5	-5	0	2	11	-17	0	5				

Lines and Pixels



Another Example

- Rasterize the line from $(3, 5)$ to $(16, 14)$.
  - $W = 13$, $H = 9$.
  - $2H = 18$, $2(W - H) = 8$, $3H - 2W = 1$.

x	F	Δy	y	x	F	Δy	y	x	F	Δy	y
3				8				13			
4				9				14			
5				10				15			
6				11				16			
7				12							

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x	F	Δy	y	x	F	Δy	y	x	F	Δy	y
3	1			8	13			13	-1		
4	-7			9	5			14	17		
5	11			10	-3			15	9		
6	3			11	15			16	1		
7	-5			12	7						

Another Example

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x	F	Δy	y	x	F	Δy	y	x	F	Δy	y
3	1	1		8	13	1		13	-1	0	
4	-7	0		9	5	1		14	17	1	
5	11	1		10	-3	0		15	9	1	
6	3	1		11	15	1		16	1	1	
7	-5	0		12	7	1					

Another Example

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  - $W = 13$, $H = 9$.
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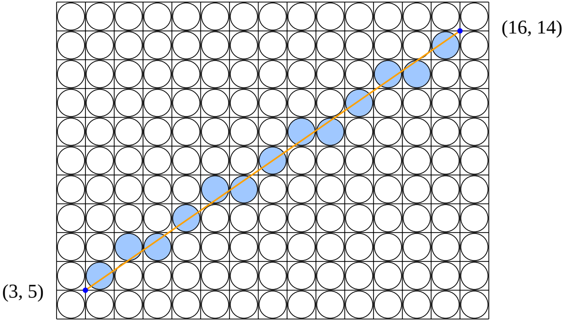
x	F	Δy	y	x	F	Δy	y	x	F	Δy	y
3	1	1	5	8	13	1	8	13	-1	0	12
4	-7	0	6	9	5	1	9	14	17	1	12
5	11	1	6	10	-3	0	10	15	9	1	13
6	3	1	7	11	15	1	10	16	1	1	14
7	-5	0	8	12	7	1	11				

Example



Bresenham's Algorithm

Bresenham's Algorithm

```
void bresenham(Point2D A, Point2D B)
{
    int y = A.y;
    int W = B.x - A.x;
    int H = B.y - A.y;
    int H2 = 2*H;
    int WH2 = 2*(W - H);
    int F = 3*H - 2*W;
    for (int x = A.x; x < B.x; x++)
    {
        setPixel(x, y);
        if (F < 0)
            F += H2;
        else
        {
            F -= WH2;
            y++;
        }
    }
}
```

Outline

Assignment

Assignment

- Rasterize the line from $(0, 0)$ to $(8, 5)$.
- Rasterize the line from $(3, 8)$ to $(22, 39)$.
- Rasterize the line from $(0, 0)$ to $(5, 8)$ by adapting the rasterization of the line from $(0, 0)$ to $(8, 5)$.
- Rasterize the line from $(0, 0)$ to $(-5, 8)$ by adapting the rasterization of the line from $(0, 0)$ to $(8, 5)$.